EXPANSION IN ONE ORTHOGONAL SYSTEM OF WATSON OPERATORS FOR SOLVING HEAT-CONDUCTION PROBLEMS

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The system of integral equations which arises in one heat-conduction problem is solved by a method based on the expansion of integral operators in an orthogonal system of Watson operators.

1. Certain heat-conduction problems can be reduced to the system of integral equations

$$\sum_{k=1}^{n} V_{ik}\varphi_k(s) + S\varphi_i(s) = Sh_i(s),\ i = 1,\ 2,\ \ldots,\ n, \tag{1}$$

where V_{ik} are the integral operators defined by

$$V_{ik}\varphi(s) = \frac{d}{ds}\left\{ s \int_{\frac{1}{s}}^{\infty} \tilde{K}_{ik}(s\sigma)\,\varphi(\sigma)\,d\sigma \right\}, \tag{2}$$

and S is the integral operator designed by

$$S\varphi(s) = \frac{1}{s}\,\varphi\left(\frac{1}{s}\right). \tag{3}$$

In particular, a system like (1) arises in the heat-conduction problem treated in [1].

Here we propose a method for solving (1) based on the expansion of the integral operators in some orthogonal system of Watson operators.

2. The Watson operator W acting in the real function space $\mathscr{L}_2(-\infty, \infty)$ is defined by

$$Wf(x) = \frac{d}{dx}\left\{ x \int_{-\infty}^{\infty} \psi^W(xs)\,f(s)\,ds \right\}. \tag{4}$$

The Watson operators were analyzed in [2, 3]. The basic results of work before 1960 are reviewed in [4].

The function $\psi^W(x)$, which we refer to below as the "kernel of the Watson operator," must satisfy the condition

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$$\int_{-\infty}^{+\infty} \psi^W(sx)\,\psi^W(tx)\,dx = \begin{cases} 0 & \text{for } st<0, \\ [\max(|s|,\ |t|)]^{-1} & \text{for } st>0. \end{cases} \tag{5}$$

We know [2, 3] that any real Watson operator W is a unitary self-adjoint operator action in $B\mathcal{L}_2(-\infty, \infty)$. It follows, in particular, that we have $W^2 = E$, where E is the identity operator.

We denote by $\mathfrak{M}_1$ the set of all real Watson operators with kernels which vanish at $x < 0$. Let us consider the function $\psi^T(x)$, which is equal to 1 for $0 \le x \le 1$ and equal to 0 for other values of the argument. We construct a linear shell of the set $\mathfrak{M}_1$. For two arbitrary operators V_1 and V_2 from this linear shell we can define a scalar product by

$$(V_1V_2) = (V_1\psi^T,\ V_2\psi^T)_{\mathcal{L}_2}. \tag{6}$$

We denote by $\{\mathfrak{M}_1\}$ a closed linear shell of the set $\mathfrak{M}_1$. $\{\mathfrak{M}_1\}$ is a Hilbert space with scalar product (6).

3. If $W_n \in \mathfrak{M}_1$ (n = 1, 2, 3, ...), then $W_1W_2W_3 \in \mathfrak{M}_1$ and, in general,

$$\prod_{k=1}^{2n+1} W_k \in \mathfrak{M}_1.$$

On the other hand, we have $W_1 \cdot W_2 \bar{\in} \mathfrak{M}_1$ and, in general,

$$\prod_{k=1}^{2n} W_k \bar{\in} \mathfrak{M}_1.$$

We denote by $\mathfrak{M}_2$ the set of products of an even number of operators from $\mathfrak{M}_1$. Constructing the linear shell of the set $\mathfrak{M}_2$, we define the scalar product of any two operators Z_1 and Z_2 from this linear shell by the equation

$$(Z_1,\ Z_2) = (Z_1\psi^T,\ Z_2\psi^T)_{\mathcal{L}_2}. \tag{7}$$

We denote by $\{\mathfrak{M}_2\}$ a closed linear shell of the set $\mathfrak{M}_2$. $\{\mathfrak{M}_2)$ is a Hilbert space with the scalar product (7).

4. The function $\psi^T(x)$ introduced above is the kernel of some Watson operators; we denote this operator by T. Let us consider the function $\psi^S(x)$, which is equal to x^{-1} for $x \ge 1$ and equal to 0 for $x < 1$. The function $\psi^S(x)$ is the kernel of some Watson operators; we denote this operator by S. It is easy to see that this operator is the same as that introduced previously by Eq. (3).

The simplest Watson operators T and S and their products play an important role in the theory below.

5. Let us consider the product of an odd number of operators T and S, taken alternately. We will list the basic properties of these operators and of certain associated operators.

A. The kernels of the operators $(TS)^nT$ with n = 0, 1, 2, ... are defined by

$$(TS)^nT\psi^T(x) = \begin{cases} (-1)^n L_n(-\ln x) & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{for other x.} \end{cases} \tag{8}$$

The kernels of the operators $(TS)^nT$ with −1, −2, −3, ... are defined by

$$(TS)^nT\psi^T(x) = \begin{cases} 0 & \text{for } x<1, \\ (-1)^{n-1}x^{-1}L_{-n-1}(\ln x) & \text{for } x \geqslant 1. \end{cases} \tag{9}$$

In (8) and (9), $L_n(z)$ is the Laguerre polynomial of degree n, defined by

$$L_n(z) = \sum_{k=0}^{n} (-1)^k \frac{n! z^k}{(n-k)!(k!)^2} \quad (n = 0,\ 1,\ 2,\ \ldots). \tag{10}$$

B. The kernels of the operators $(TS)^nT$ with n = 0, 1, 2, ... form a complete orthonormal system in $\mathcal{L}_2(0, 1)$, and those with n = -1, -2, -3, ... form a complete orthonormal system in $\mathcal{L}_2(1, \infty)$. The set of all these kernels forms a complete orthonormal system in $\mathcal{L}_2(0, \infty)$.

C. If $W \in \mathfrak{M}_1$, then $(TS)^nW$ (n = 0, ±1, ±2, ...) is a complete orthogonal system in $\{\in \mathfrak{M}_1\}$. In particular, in the simplest case, $(TS)^nT$ (n = 0, ±1, ±2, ...) is a complete orthogonal system in $\{\mathfrak{M}_1\}$.

D. If W_1 and W_2 are arbitrary operators from $\{\mathfrak{M}_1\}$, then $W_2(TS)^nW_1$ (n = 0, ±1, ±2, ...) is a complete orthogonal system in $\{\mathfrak{M}_2\}$. In particular, taking $W_2 = T$, $W_1 = S$, we find $(TS)^n$ (n = 0, ±1, ±2, ...$\{\mathfrak{M}_1\}$, complete orthogonal system in $\{\{\mathfrak{M}_2)$.

E. An arbitrary operator $V \in \{\mathfrak{M}_1\}$ can be written in a unique manner by

$$V = \sum_{n=-\infty}^{+\infty} a_n (TS)^n T. \tag{11}$$

If V is defined by

$$Vf(x) = \frac{d}{dx} \left\{ x \int_{-\infty}^{+\infty} \tilde{K}(xs) f(s)\, ds \right\}, \tag{12}$$

where the kernel $\tilde{K}(x)$ vanishes for x < 0, then in expansion (11) the coefficients a_n (n = 0, ±1, ±2, ...) can be found from

$$a_n = (-1)^n \int_0^1 \tilde{K}(x) L_n(-\ln x)\, dx \quad (n = 0,\ 1,\ 2,\ \ldots) \tag{13}$$

and

$$a_n = (-1)^{n-1} \int_1^{\infty} \tilde{K}(x) x^{-1} L_{-n-1}(\ln x)\, dx \quad (n = -1,\ -2,\ \ldots). \tag{14}$$

Here we are assuming $\tilde{K}(x) \in \mathcal{L}_2(0, \infty)$.

F. An arbitrary operator Z from $\{\mathfrak{M}_2\}$, represented by the expansions

$$Z = \sum_{n=-\infty}^{+\infty} a_n (TS)^n. \tag{15}$$

6. Let us consider the subspace of operators from $\{\mathfrak{M}_2\}$, represented by the expansions

$$Z = \sum_{n=0}^{\infty} a_n (TS)^n. \tag{16}$$

We can operate on the operator series in (16) by analogy with the operations on ordinary power series, since we obviously have $(TS)^m(TS)^n = (TS)^{m+n}$. In particular, the coefficients b_n (n = 0, 1, 2, ...) of the series $\sum_{n=0}^{\infty} b_n (TS)^n$ can always be determined in a unique manner from the condition

$$\left(\sum_{n=0}^{\infty} a_n (TS)^n\right)\left(\sum_{n=0}^{\infty} b_n (TS)^n\right) = E = (TS)^0, \tag{17}$$

provided that $a_0 \neq 0$. If the coefficients b_n ($n = 0, 1, 2, \ldots$) which are found determine an operator from $\{\mathfrak{W}_2\}$, this operator is the inverse of Z:

$$Z^{-1}=\sum_{n=0}^{\infty} b_n (TS)^n. \tag{18}$$

All this discussion can be repeated for the operators from $\{\mathfrak{W}_2\}$, determined by expansions of the type $\sum_{n=0}^{\infty} a_n (ST)^n$.

7. The system of integral equations derived in [1] was converted to the form

$$\begin{aligned}
&-(a\alpha_1 V_1+S)\varphi_1(s)+(V_3-\alpha_1 V_2)\varphi_2(s)+b\beta_1 V_1\varphi_3(s)+\beta_1 V_4\varphi_4(s)=Sh_1(s),\\
&a\beta_1 V_1\varphi_1(s)+\beta_2 V_2\varphi_2(s)-(b\alpha_2 V_1+S)\varphi_3(s)+(V_5-\alpha_2 V_4)\varphi_4(s)=Sh_2(s),\\
&(V_3-\alpha_3 V_2)\varphi_1(s)-(a\alpha_3 V_1+S)\varphi_2(s)+\beta_3 V_4\varphi_3(s)+b\beta_1 V_1\varphi_4(s)=Sh_3(s),\\
&\beta_4 V_2\varphi_1(s)+a\beta_4 V_1\varphi_2(s)+(V_5-\alpha_4 V_4)\varphi_3(s)-(b\alpha_4 V_1+S)\varphi_4(s)=Sh_4(s),
\end{aligned} \tag{19}$$

where α_i, β_i ($i = 1, 2, 3, 4$), a, b are constants; $h_i(s)$ ($i = 1, 2, 3, 4$) are given functions; and $\varphi_i(s)$ ($i = 1, 2, 3, 4$) are unknown functions. Each of the operators V_k ($k = 1, 2, 3, 4, 5$) which appear in (19) satisfies

$$V_i\varphi(s) = \frac{d}{ds}\left\{s\int_{\frac{1}{s}}^{\infty}\tilde{K}_i(s\delta)\,\varphi(\delta)\,d\delta\right\} \quad (i=1,\ 2,\ 3,\ 4,\ 5), \tag{20}$$

where the kernels $\tilde{K}_i(x)$, determined for $x \geq 1$, are given by

$$\tilde{K}_1(x) = \frac{2\sqrt{\ln x}}{\sqrt{\pi}\,x}, \tag{21}$$

$$\tilde{K}_2(x)=\frac{2a\sqrt{\ln x}}{\sqrt{\pi}\,x}\exp\left(-\frac{l_1^2}{4a^2\ln x}\right)-\frac{l_1}{x}\operatorname{erfc}\left(\frac{l_1}{2a\sqrt{\ln x}}\right), \tag{22}$$

$$\tilde{K}_3(x) = \frac{2}{x}\operatorname{erfc}\left(\frac{l_1}{4a\sqrt{\ln x}}\right), \tag{23}$$

$$\tilde{K}_4(x) = \frac{2b\sqrt{\ln x}}{\sqrt{\pi}\,x}\exp\left(-\frac{l_2^2}{4b^2\ln x}\right)-\frac{l_2}{x}\operatorname{erfc}\left(\frac{l_2}{2b\sqrt{\ln x}}\right), \tag{24}$$

$$\tilde{K}_5(x) = \frac{2}{x}\operatorname{erfc}\left(\frac{l_2}{2b\sqrt{\ln x}}\right). \tag{25}$$

We expand each of the kernels in (21)-(25) in a complete orthonormal (for $x \geq 1$) system of function $x^{-1}L_n(\ln x)$. This expansion can be carried out, since each of the kernels in (21)-(25) belongs to the function space $\mathscr{L}_2(1, \infty)$. We find the following expansions:

$$\tilde{K}_i(x) = \sum_{n=0}^{\infty} a_n^i x^{-1} L_n(\ln x),\quad i=1,\ 2,\ 3,\ 4,\ 5. \tag{26}$$

The coefficients a_n^i of these expansion are

$$a_n^i = \int_1^{\infty}\tilde{K}_i(x)\,x^{-1}L_n(\ln x)\,dx \quad (n=1,\ 2,\ 3,\ \ldots;\ i=1,\ 2,\ 3,\ 4,\ 5). \tag{27}$$

We must resort to numerical methods to evaluate the integrals on the right side of (27) for i = 1, 2, 3, 4, 5. In the case i = 1 the integral in (27) turns out to be elementary, and calculations yield

$$a_n^1 = \sum_{k=0}^{n} \frac{(-1)^k n!\,(2k+1)!!}{(n-k)!\,(k!)^2 2^k}. \tag{28}$$

Knowing that the kernels of the operators $S(TS)^n$ (n = 0, 1, 2, ...) are given by

$$S(TS)^n \psi^T(x) = \begin{cases} (-1)^n x^{-1} L_n(\ln x) & \text{for } x \geqslant 1, \\ 0 & \text{for } x < 1, \end{cases} \tag{29}$$

which follows from (9), we can write the expansions

$$V_i = \sum_{n=0}^{\infty} (-1)^n a_n^i S(TS)^n, \quad i = 1, 2, 3, 4, 5. \tag{30}$$

We denote by V_{ik} the integral operator acting on the k-th unknown function in the i-th equation of system (19) (i, k = 1, 2, 3, 4).

We thus have

$$\begin{gathered} V_{11} = a\alpha_1 V_1 - S;\ V_{12} = V_3 - \alpha_1 V_2;\ V_{13} = b\beta_1 V_1;\ V_{14} = \beta_1 V_4; \\ V_{21} = a\beta_1 V_1;\ V_{22} = \beta_2 V_2;\ V_{23} = -b\alpha_2 V_1 - S;\ V_{24} = V_5 - \alpha_2 V_4; \\ V_{31} = V_3 - \alpha_3 V_2;\ V_{32} = -a\alpha_3 V_1 - S;\ V_{33} = \beta_3 V_4;\ V_{34} = b\beta_3 V_1; \\ V_{41} = \beta_4 V_2;\ V_{42} = a\beta_4 V_1;\ V_{43} = V_5 - \alpha_4 V_4;\ V_{44} = -b\alpha_4 V_1 - S. \end{gathered} \tag{31}$$

Each of the operators V_{ik} (i, k = 1, 2, 3, 4) can be written as an expansion in orthogonal Watson operators:

$$V_{ik} = \sum_{n=0}^{\infty} a_n^{ik} S(TS)^n \quad (i,\ k = 1, 2, 3, 4), \tag{32}$$

where the coefficients a_n^{ik} are expressed in terms of a_n^i and in terms of the constants, α_i, β_i (i = 1, 2, 3, 4), a, b which appear in the original system.

System (19) can now be written in the form

$$\sum_{k=1}^{4} V_{ik} \varphi_k(s) = Sh_k(s) \quad (i = 1, 2, 3, 4). \tag{33}$$

8. We now transform system (33), all operators in which belong to the space $\{\mathfrak{W}_1\}$, into a new system, whose operators belong to the space $\{\mathfrak{W}_2\}$. The purpose of this transformation is to obtain a system all operators of which are commutative and can be expanded in power operator series in terms of nonnegative powers of the operator TS. For such a transformation of system (23) it is sufficient to apply the operator S to the left and right sides of all equations of this system. Using the notation $Z_{ik} = SV_{ik}$ (i, k = 1, 2, 3, 4), we can write the operators Z_{ik} as

$$Z_{ik} = \sum_{n=0}^{\infty} a_n^{ik} (TS)^n \quad (i,\ k = 1, 2, 3, 4). \tag{34}$$

System (33) becomes

$$\sum_{k=1}^{4} Z_{ik}\varphi_k(s) = h_k(s) \quad (i = 1,\ 2,\ 3,\ 4). \tag{35}$$

Clearly, since the operators Z_{ik} are commutative, we can deal with them as we would deal with coefficients of an ordinary algebraic system of linear equations. In speaking of division by an operator, of course, we mean multiplication by the inverse operator. We introduce the operator $\Delta \in \{\mathfrak{W}_2\}$, which serves as the determinant of the system of linear equations:

$$\Delta = \operatorname{Det} Z_{ik}. \tag{36}$$

Let us find the operators A_{ik} = (i, k = 1, 2, 3, 4) which are the signed cofactors of the elements Z_{ik} in the determinant of system (35); clearly we have $A_{ik} \in \{\mathfrak{W}_2\}$ (i, k = 1, 2, 3, 4). We write the solution of (35) in the form

$$\varphi_k(s) = \Delta^{-1} \sum_{i=1}^{4} A_{ki} h_i(s),\ k = 1,\ 2,\ 3,\ 4. \tag{37}$$

Recalling that in [1] the unknown intensities of the thermal potentials $\rho_k(t)$ were related to the function $\varphi_k(s)$ by

$$\varphi_k(s) = \rho_k(-\ln s), \quad k = 1,\ 2,\ 3,\ 4 \tag{38}$$

and that

$$h_k(s) = 2\Phi_k(-\ln s), \quad k = 1,\ 2,\ 3,\ 4, \tag{39}$$

where $\Phi_i(t)$ are the original, given functions of the time, we can express the unknown intensities of the thermal potentials in the following manner:

$$\rho_k(t) = \varphi_k(\exp(-t)),\ k = 1,\ 2,\ 3,\ 4, \tag{40}$$

where

$$\varphi_k(s) = \Delta^{-1} \sum_{i=1}^{4} A_{ki} 2\Phi_i(-\ln s),\ k = 1,\ 2,\ 3,\ 4. \tag{41}$$

Using the functions $\varphi_k(s)$ found, we can write the solution of the problem posed in [1] in the following form

$$\begin{aligned} u(x,\ t) = \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{t} (t-\tau)^{-\frac{1}{2}} \Big\{ &\varphi_1[\exp(-\tau)] \exp\left(-\frac{x^2}{4a^2(t-\tau)}\right) + \\ &+ \varphi_2[\exp(-\tau)] \exp\left(-\frac{(x-l_1)^2}{4a^2(t-\tau)}\right)\Big\} d\tau, \end{aligned} \tag{42}$$

$$\begin{aligned} v(y,\ t) = \frac{b}{2\sqrt{\pi}} \int_{-\infty}^{t} (t-\tau)^{-\frac{1}{2}} \Big\{ &\varphi_3[\exp(-\tau)] \exp\left(-\frac{y^2}{4b^2(t-\tau)}\right) + \\ &+ \varphi_4[\exp(-\tau)] \exp\left(-\frac{(y-l_2)^2}{4b^2(t-\tau)}\right)\Big\} d\tau. \end{aligned} \tag{43}$$

9. Each of the functions $\varphi_k(s)$ (k = 1, 2, 3, 4) is calculated by applying the operator series $\sum_{n=0}^{\infty} b_n (TS)^n$ to the given function $h_k(s)$. The actual calculation of the values of

the function $\varphi_k(s)$ can be carried out without resorting to tables; the result of the operation with operator TS on an arbitrary function f(x) is

$$TSf(x) = \int_x^\infty \frac{f(s)}{s} ds - f(x). \tag{44}$$

Accordingly, the terms of the series $\sum_{n=0}^{\infty} b_n (TS)^n f(x)$ can be found successively from

$$TSf(x) = \int_x^\infty \frac{f(s)}{s} ds - f(x),$$

$$(TS)^2 f(x) = \int_x^\infty \frac{TSf(s)}{s} ds - TSf(x),$$

$$(TS)^3 f(x) = \int_x^\infty \frac{(TS)^2 f(s)}{s} ds - (TS)^2 f(x),$$

etc.

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